

# Multiple positive solutions for $(n-1, 1)$ -type semipositone conjugate boundary value problems of nonlinear fractional differential equations\*

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**Abstract.** In this paper, we consider  $(n-1, 1)$ -type conjugate boundary value problem for the nonlinear fractional differential equation

$$\begin{aligned} \mathbf{D}_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) &= 0, \quad 0 < t < 1, \lambda > 0, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ u(1) &= 0, \end{aligned}$$

where  $\lambda$  is a parameter,  $\alpha \in (n-1, n]$  is a real number and  $n \geq 3$ , and  $\mathbf{D}_{0+}^{\alpha}$  is the Riemann-Liouville's fractional derivative, and  $f$  is continuous and semipositone. We give properties of Green's function of the boundary value problems, and derive an interval of  $\lambda$  such that any  $\lambda$  lying in this interval, the semipositone boundary value problem has multiple positive solutions.

**Key words.** Riemann-Liouville's fractional derivative; fractional differential equation; boundary value problem; positive solution; fractional Green's function; fixed-point theorem.

**MR(2000) Subject Classifications:** 34B15

## 1 Introduction

We consider the  $(n-1, 1)$ -type conjugate boundary value problem of nonlinear fractional differential equation involving Riemann-Liouville's derivative

$$\begin{aligned} \mathbf{D}_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) &= 0, \quad 0 < t < 1, \lambda > 0, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ u(1) &= 0, \end{aligned} \tag{1.1}$$

where  $\lambda$  is a parameter,  $\alpha \in (n-1, n]$  is a real number and  $n \geq 3$ ,  $\mathbf{D}_{0+}^{\alpha}$  is the Riemann-Liouville's fractional derivative,  $f : (0, 1) \times [0, +\infty) \rightarrow (-\infty, +\infty)$  is a sign-changing continuous function. As far as we know, there are few papers which deal with the boundary value problem for nonlinear fractional differential equation.

Because of fractional differential equation's modeling capabilities in engineering, science, economy, and other fields, the last few decades has resulted in a rapid development of the theory of fractional differential equation; see [1]-[7] for a good overview. Within this development, a fair amount of the theory has been devoted to initial and boundary value problems (see [9]-[20]). In most papers, the definition of fractional derivative is the Riemann-Liouville's fractional derivative or the Caputo's fractional derivative. For details see references.

In this paper, we give sufficient conditions for the existence of positive solution of the semipositone boundary value problems (1.1) for a sufficiently small  $\lambda > 0$  where  $f$  may change sign. Our analysis relies on nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems.

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## 2 Preliminaries

For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of Riemann-Liouville's derivatives of fractional order which can be found in [3].

**Definition 1.1** [3] The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0$$

where  $\alpha > 0$ , is called Riemann-Liouville fractional integral of order  $\alpha$ .

**Definition 1.2** [3] For a function  $f(x)$  given in the interval  $[0, \infty)$ , the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of number  $\alpha$ , is called the Riemann-Liouville fractional derivative of order  $s$ .

From the definition of Riemann-Liouville's derivative, we can obtain the statement.

As examples, for  $\mu > -1$ , we have

$$\mathbf{D}_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

giving in particular  $\mathbf{D}_{0+}^{\alpha} x^{\alpha-m}$ ,  $m = i, 2, 3, \dots, N$ , where  $N$  is the smallest integer greater than or equal to  $\alpha$ .

**Lemma 2.1** Let  $\alpha > 0$ , then the differential equation

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0$$

has solutions  $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$ ,  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ , as unique solutions, where  $n$  is the smallest integer greater than or equal to  $\alpha$ .

As  $\mathbf{D}_{0+}^{\alpha} I_{0+}^{\alpha} u = u$  From the lemma 2.1, we deduce the following statement.

**Lemma 2.2** Let  $\alpha > 0$ , then

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

for some  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ ,  $n$  is the smallest integer greater than or equal to  $\alpha$ .

The following a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems, will play major role in our next analysis.

**Theorem 2.3** ([12]) Let  $X$  be a Banach space with  $\Omega \subset X$  be closed and convex. Assume  $U$  is a relatively open subsets of  $\Omega$  with  $0 \in U$ , and let  $S : \overline{U} \rightarrow \Omega$  be a compact, continuous map. Then either

1.  $S$  has a fixed point in  $\overline{U}$ , or
2. there exists  $u \in \partial U$  and  $\nu \in (0, 1)$ , with  $u = \nu Su$ .

**Theorem 2.4** ([8]) Let  $X$  be a Banach space, and let  $P \subset X$  be a cone in  $X$ . Assume  $\Omega_1, \Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ , and let  $S : P \rightarrow P$  be a completely continuous operator such that, either

1.  $\|Sw\| \leq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_2$ , or
2.  $\|Sw\| \geq \|w\|$ ,  $w \in P \cap \partial\Omega_1$ ,  $\|Sw\| \leq \|w\|$   $w \in P \cap \partial\Omega_2$ .

Then  $S$  has a fixed point in  $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .

### 3 Green's Function and Its Properties

In this section, we derive the corresponding Green's function for boundary value problem (1.1), and obtained some properties of the Green's function. First of all, we find the Green's function for boundary-value problem (1.1).

**Lemma 3.1** *Let  $h(t) \in C[0, 1]$  be a given function, then the boundary-value problem*

$$\begin{aligned} \mathbf{D}_{0+}^{\alpha} u(t) + h(t) &= 0, \quad 0 < t < 1, 2 \leq n-1 < \alpha \leq n, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ u(1) &= 0 \end{aligned} \quad (3.1)$$

*has a unique solution*

$$u(t) = \int_0^1 G(t, s) h(s) ds, \quad (3.2)$$

*where*

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.3)$$

*Here  $G(t, s)$  is called the Green's function of boundary value problem (3.1).*

**Proof** By means of the Lemma2.2, we can reduce (3.1) to an equivalent integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds.$$

From  $u^{(j)}(0) = 0, 0 \leq j \leq n-2$ , we have  $c_j = 0, 2 \leq j \leq n$ . Then

$$u(t) = c_1 t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds.$$

From  $u(1) = 0$ , we have  $c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds$ . Then, the unique solution of (3.1) is

$$u(t) = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds = \int_0^1 G(t, s) h(s) ds.$$

**Lemma 3.2** *The Green's function  $G(t, s)$  defined by (3.3) has the following properties:*

(R1)  $G(t, s) = G(1-s, 1-t)$ , for  $t, s \in [0, 1]$ ,

(R2)  $\Gamma(\alpha)k(t)q(s) \leq G(t, s) \leq (\alpha-1)q(s)$ , for  $t, s \in [0, 1]$ ,

(R3)  $\Gamma(\alpha)k(t)q(s) \leq G(t, s) \leq (\alpha-1)k(t)$ , for  $t, s \in [0, 1]$ ,

*where*

$$k(t) = \frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha)}, \quad q(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \quad (3.4)$$

**Proof** (R1) From the definition of  $G(t, s)$ , it is obviously that  $G(t, s) = G(1-s, 1-t)$ , for  $t, s \in (0, 1)$ .

(R2) For  $s \leq t$ , we have  $1-s \geq 1-t$ , then

$$\begin{aligned} \Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (\alpha-1) \int_{t-s}^{t-ts} x^{\alpha-2} dx \\ &\leq (\alpha-1)(t-ts)^{\alpha-2}((t-ts) - (t-s)) \\ &= (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s \\ &\leq (\alpha-1)s(1-s)^{\alpha-1} \\ &= \Gamma(\alpha)(\alpha-1)q(s) \end{aligned}$$

and

$$\begin{aligned}
\Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (t-ts)^{\alpha-2}(t-ts) - (t-s)^{\alpha-2}(t-s) \\
&\geq (t-ts)^{\alpha-2}(t-ts) - (t-ts)^{\alpha-2}(t-s) \\
&= t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s \\
&\geq t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \\
&= \Gamma^2(\alpha)k(t)q(s).
\end{aligned}$$

For  $t \leq s$ , since  $\alpha > 2$ , we have

$$\begin{aligned}
\Gamma(\alpha)G(t, s) &= t^{\alpha-1}(1-s)^{\alpha-1} \leq (\alpha-1)t^{\alpha-2}t(1-s)^{\alpha-1} \\
&\leq (\alpha-1)t^{\alpha-2}s(1-s)^{\alpha-1} \leq (\alpha-1)s(1-s)^{\alpha-1} \\
&= \Gamma(\alpha)(\alpha-1)q(s)
\end{aligned}$$

and

$$\Gamma(\alpha)G(t, s) = t^{\alpha-1}(1-s)^{\alpha-1} \geq t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} = \Gamma^2(\alpha)k(t)q(s).$$

Thus  $\Gamma(\alpha)k(t)q(s) \leq G(t, s) \leq \Gamma(\alpha)(\alpha-1)q(s)$ , for  $t, s \in (0, 1)$ .

(R3) From (R1) and (R2), we have

$$\Gamma(\alpha)k(t)q(s) \leq G(t, s) = G(1-s, 1-t) \leq (\alpha-1)q(1-t) = (\alpha-1)k(t).$$

This completes the proof.

## 4 Main Results

We make the following assumptions:

(H<sub>1</sub>)  $f(t, u) \in C([0, 1] \times [0, +\infty), (-\infty, +\infty))$ , moreover there exists a function  $g(t) \in L^1([0, 1], (0, +\infty))$  such that  $f(t, u) \geq -g(t)$ , for any  $t \in (0, 1)$ ,  $u \in [0, +\infty)$ .

(H<sub>1</sub><sup>\*</sup>)  $f(t, u) \in C((0, 1) \times [0, +\infty), (-\infty, +\infty))$  may be singular at  $t = 0, 1$ , moreover there exists a function  $g(t) \in L^1((0, 1), (0, +\infty))$  such that  $f(t, u) \geq -g(t)$ , for any  $t \in (0, 1)$ ,  $u \in [0, +\infty)$ .

(H<sub>2</sub>)  $f(t, 0) > 0$ , for any  $t \in [0, 1]$ .

(H<sub>3</sub>) There exists  $[\theta_1, \theta_2] \in (0, 1)$  such that  $\lim_{u \uparrow +\infty} \min_{t \in [\theta_1, \theta_2]} \frac{f(t, u)}{u} = +\infty$ .

(H<sub>4</sub>)  $0 < \int_0^1 q(s)g(s)ds < +\infty$  and  $\int_0^1 q(s)f(s, y)ds < +\infty$  for any  $y \in [0, R]$ ,  $R > 0$  is any constant.

In fact, we only consider the boundary value problem

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}x(t) + \lambda[f(t, [x(t) - v(t)]^*) + g(t)] = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, \lambda > 0, \\
&x^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \\
&x(1) = 0
\end{aligned} \tag{4.1}$$

where

$$y(t)^* = \begin{cases} y(t), & y(t) \geq 0; \\ 0, & y(t) < 0. \end{cases}$$

and  $v(t) = \lambda \int_0^1 G(t, s)g(s)ds$ , which is the solution of the boundary value problem

$$\begin{aligned}
&-\mathbf{D}_{0+}^{\alpha}v(t) = \lambda g(t), \quad 0 < t < 1, n-1 < \alpha \leq n, \lambda > 0, \\
&v^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \\
&v(1) = 0.
\end{aligned}$$

We will show there exists a solution  $x$  for the boundary value problem (4.1) with  $x(t) \geq v(t), t \in [0, 1]$ . If this is true, then  $u(t) = x(t) - v(t)$  is a nonnegative solution (positive on  $(0, 1)$ ) of the boundary value problem (1.1). Since for any  $t \in (0, 1)$ ,

$$-\mathbf{D}_{0+}^{\alpha}u - \mathbf{D}_{0+}^{\alpha}v = \lambda[f(t, u) + g(t)],$$

we have

$$-\mathbf{D}_{0+}^{\alpha}u = \lambda f(t, u).$$

As a result, we will concentrate our study on the boundary value problem (4.1).

We note that  $x(t)$  is a solution of (4.1) if and only if

$$x(t) = \lambda \int_0^1 G(t, s)(f(s, [x(s) - v(s)]^*) + g(s))ds, \quad 0 \leq t \leq 1. \quad (4.2)$$

For our constructions, we shall consider the Banach space  $E = C[0, 1]$  equipped with standard norm  $\|x\| = \max_{0 \leq t \leq 1} |x(t)|, x \in X$ . We define a cone  $P$  by

$$P = \{x \in X | x(t) \geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} \|x\|, \quad t \in [0, 1], \alpha \in (n-1, n], n \geq 3\}.$$

**Lemma 4.1** Assume  $(H_1)$  (or  $(H_1^*)$ ) is satisfied and define the integral operator  $T : P \rightarrow E$  by

$$Tx(t) = \lambda \int_0^1 G(t, s)(f(s, [x(s) - v(s)]^*) + g(s))ds, \quad 0 \leq t \leq 1, x \in P. \quad (4.3)$$

Then  $T : P \rightarrow P$  is completely continuous.

**Proof** First, we prove that  $T : P \rightarrow P$ .

Notice from (4.3) and Lemma 3.2 that, for  $x \in P$ ,  $Tx(t) \geq 0$  on  $[0, 1]$  and

$$\begin{aligned} Tx(t) &= \lambda \int_0^1 G(t, s)(f(s, [x(s) - v(s)]^*) + g(s))ds \\ &\leq \lambda \int_0^1 (\alpha-1)q(s)(f(s, [x(s) - v(s)]^*) + g(s))ds, \end{aligned}$$

then  $\|Tx\| \leq \int_0^1 (\alpha-1)q(s)(f(s, [x(s) - v(s)]^*) + g(s))ds$ .

On the other hand, we have

$$\begin{aligned} Tx(t) &= \lambda \int_0^1 G(t, s)(f(s, [x(s) - v(s)]^*) + g(s))ds \\ &\geq \lambda \int_0^1 t^{\alpha-2}(1-t)q(s)(f(s, [x(s) - v(s)]^*) + g(s))ds \\ &\geq \frac{t^{\alpha-2}(1-t)}{\alpha-1} \lambda \int_0^1 (\alpha-1)q(s)(f(s, [x(s) - v(s)]^*) + g(s))ds \\ &\geq \frac{t^{\alpha-2}(1-t)}{\alpha-1} \|Tx\|. \end{aligned}$$

Thus,  $T(P) \subset P$ . In addition, from  $f$  is continuous it follows that  $T$  is continuous.

Next, we show  $T$  is uniformly bounded.

Let  $D \subset P$  be bounded, i.e. there exists a positive constant  $L > 0$  such that  $\|y\| \leq L$ , for all  $y \in D$ . Let  $M = \max_{0 \leq t \leq 1, 0 \leq y \leq L} |f(t, y) + e(t)| + 1$ , then for  $x \in D$ , from the Lemma 3.1, one has

$$\begin{aligned} |Ty(t)| &\leq \int_0^1 |G(t, s)(f(s, [x(s) - v(s)]^*) + g(s))|ds \\ &\leq \int_0^1 |(\alpha-1)q(s)(f(s, [x(s) - v(s)]^*) + g(s))|ds \\ &\leq (\alpha-1)M \int_0^1 q(s)ds \end{aligned}$$

Hence,  $T(D)$  is bounded.

Finally, we show  $T$  is equicontinuous.

For all  $\varepsilon > 0$ , each  $u \in P$ ,  $t_1, t_2 \in [0, 1]$ ,  $t_1 < t_2$ , let

$$\eta = \min\left\{\frac{1}{2}, \frac{\Gamma(\alpha)\varepsilon}{M\alpha}\right\},$$

we will prove that  $|Tu(t_2) - Tu(t_1)| < \varepsilon$ , when  $t_2 - t_1 < \eta$ . One has

$$\begin{aligned} & |Tu(t_2) - Tu(t_1)| \\ &= \left| \int_0^1 G(t_2, s)(f(s, [x(s) - v(s)]^*) + g(s))ds - \int_0^1 G(t_1, s)(f(s, [x(s) - v(s)]^*) + g(s))ds \right| \\ &\leq \int_0^1 |G(t_2, s) - G(t_1, s)|(f(s, [x(s) - v(s)]^*) + g(s))|ds \\ &\leq M \int_0^1 |G(t_2, s) - G(t_1, s)|ds \\ &\leq M \left( \int_0^{t_1} |(G(t_2, s) - G(t_1, s))|ds + \int_{t_1}^{t_2} |(G(t_2, s) - G(t_1, s))|ds + \int_{t_1}^{t_2} |(G(t_2, s) - G(t_1, s))|ds \right) \\ &\leq M \left( \int_0^{t_1} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-1} + ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})}{\Gamma(\alpha)} ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\ &\leq M \left( \int_0^1 \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t_1} \frac{-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\ &\leq \frac{M}{\Gamma(\alpha)} ((\alpha-1)t_2^{\alpha-2}\eta \int_0^1 (1-s)^{\alpha-1} ds + \frac{1}{\alpha}(t_2^\alpha - t_1^\alpha)) \\ &\leq \frac{M}{\Gamma(\alpha)} ((\alpha-1)t_2^{\alpha-2}\eta \int_0^1 (1-s)^{\alpha-1} ds + t_2^{\alpha-1}\eta) \\ &\leq \frac{M}{\Gamma(\alpha)} ((\alpha-1) \int_0^1 (1-s)^{\alpha-1} ds + 1)\eta \\ &\leq \frac{M\alpha\eta}{\Gamma(\alpha)} < \varepsilon. \end{aligned}$$

Thus, we obtain

$$|Tu(t_2) - Tu(t_1)| < \frac{M\alpha\eta}{\Gamma(\alpha)} < \varepsilon.$$

By means of the Arzela-Ascoli theorem,  $T : P \rightarrow P$  is completely continuous.

If condition  $(H_1)$  is replaced by  $(H_1^*)$ , let

$$T_n x(t) = \lambda \int_0^1 G(t, s)(f_n^*(s, [x(s) - v(s)]^*) + g(s))ds, \quad n \geq 2,$$

where

$$f_n^*(t, y) = \begin{cases} \inf\{f(t, y), f(\frac{1}{n}, y)\}, & 0 < t \leq \frac{1}{n}; \\ f(t, y), & \frac{1}{n} \leq t \leq \frac{n-1}{n}; \\ \inf\{f(t, y), f(\frac{n-1}{n}, y)\}, & \frac{n-1}{n} \leq t < 1. \end{cases}$$

It is easy to see that  $f_n^*(t, y) \in C([0, 1] \times [0, \infty))$  is bounded and  $0 \leq f_n^*(t, y) \leq f(t, y)$ ,  $t \in (0, 1)$ . By repeating the similar proof above, we get that  $T_n$  is a completely continuous operator on  $P$  for each  $n \geq 2$ . Furthermore, for any  $R > 0$ , set  $\Omega_R = \{u \in P : \|u\| \leq R\}$ , then  $T_n$  converges uniformly to  $T$  on  $\Omega_n$  as  $n \rightarrow \infty$ . In fact, for  $R > 0$  and

$u \in \Omega_R$ , we have

$$\begin{aligned}
& |T_n u(t) - Tu(t)| \\
&= \left| \int_0^1 G(t,s) (f_n^*(s, [x(s) - v(s)]^*) - f(s, [x(s) - v(s)]^*)) ds \right| \\
&\leq \int_0^1 (\alpha - 1) q(s) |f_n^*(s, [x(s) - v(s)]^*) - f(s, [x(s) - v(s)]^*)| ds \\
&= \int_0^{\frac{1}{n}} (\alpha - 1) q(s) |f_n^*(s, [x(s) - v(s)]^*) - f(s, [x(s) - v(s)]^*)| ds \\
&\quad + \int_{\frac{n-1}{n}}^1 (\alpha - 1) q(s) |f_n^*(s, [x(s) - v(s)]^*) - f(s, [x(s) - v(s)]^*)| ds \\
&\leq \int_0^{\frac{1}{n}} (\alpha - 1) q(s) \max_{0 \leq y \leq R} |f_n^*(s, y) - f(s, y)| ds \\
&\quad + \int_{\frac{n-1}{n}}^1 (\alpha - 1) q(s) \max_{0 \leq y \leq R} |f_n^*(s, y) - f(s, y)| ds \rightarrow 0 \quad (n \rightarrow \infty).
\end{aligned}$$

So we conclude that  $T_n$  converges uniformly to  $T$  on  $\Omega_n$  as  $n \rightarrow \infty$ . Thus,  $T$  is completely continuous.

The proof is completed.

**Theorem 4.2** Suppose that  $(H_1)$ – $(H_2)$  hold. Then there exists a constant  $\bar{\lambda} > 0$  such that, for any  $0 < \lambda \leq \bar{\lambda}$ , the boundary value problem (1.1) has at least one positive solutions.

**Proof** Fixed  $\delta \in (0, 1)$ , from  $(H_2)$ , Let  $0 < \varepsilon < 1$  be such that

$$f(t, z) \geq \delta f(t, 0), \quad \text{for } 0 \leq t \leq 1, \quad 0 \leq z \leq \varepsilon. \quad (4.4)$$

Suppose

$$0 < \lambda < \frac{\varepsilon}{2c\bar{f}(\varepsilon)} := \bar{\lambda}$$

where  $\bar{f}(\varepsilon) = \max_{0 \leq t \leq 1, 0 \leq z \leq \varepsilon} \{f(t, z) + g(t)\}$  and  $c = \int_0^1 (\alpha - 1) q(s) ds$ . Since

$$\lim_{z \downarrow 0} \frac{\bar{f}(z)}{z} = +\infty$$

and

$$\frac{\bar{f}(\varepsilon)}{\varepsilon} < \frac{1}{2c\bar{\lambda}},$$

then exist a  $R_0 \in (0, \varepsilon)$  such that

$$\frac{\bar{f}(R_0)}{R_0} = \frac{1}{2c\bar{\lambda}}.$$

Let  $x \in P$  and  $\nu \in (0, 1)$  be such that  $x = \nu T(x)$ , we claim that  $\|x\| \neq R_0$ . In fact, if  $\|x\| = R_0$ , then

$$\begin{aligned}
\|x\| &= \nu \|Tx\| \leq \nu \lambda \int_0^1 (\alpha - 1) q(s) [f(s, [x(s) - v(s)]^*) + g(s)] ds \\
&\leq \lambda \int_0^1 (\alpha - 1) q(s) [f(s, [x(s) - v(s)]^*) + g(s)] ds \\
&\leq \lambda \int_0^1 (\alpha - 1) q(s) \max_{0 \leq s \leq 1, 0 \leq z \leq R_0} [f(s, z) + g(s)] ds \\
&\leq \lambda \int_0^1 (\alpha - 1) q(s) \bar{f}(R_0) ds \\
&\leq \lambda c \bar{f}(R_0),
\end{aligned}$$

that is

$$\frac{\bar{f}(R_0)}{R_0} \geq \frac{1}{c\lambda} > \frac{1}{2c\lambda} = \frac{\bar{f}(R_0)}{R_0}$$

which implies that  $\|x\| \neq R_0$ . Let  $U = \{x \in P : \|x\| < R_0\}$ . By the nonlinear alternative theorem of Leray-Schauder type,  $T$  has a fixed point  $x \in \bar{U}$ . Moreover, if we combine (4.3), (4.4) and  $R_0 < \varepsilon$ , we obtain

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t, s) [f(s, [x(s) - v(s)]^*) + g(s)] ds \\ &\geq \lambda \int_0^1 G(t, s) [\delta f(s, 0) + g(s)] ds \\ &\geq \lambda [\delta \int_0^1 G(t, s) f(s, 0) ds + \int_0^1 G(t, s) g(s) ds] \\ &> \lambda \int_0^1 G(t, s) g(s) ds \\ &= v(t) \quad \text{for } t \in (0, 1). \end{aligned}$$

Let  $u(t) = x(t) - v(t) > 0$ . Then (1.1) has a positive solution  $u$  and  $\|u\| \leq \|x\| \leq R_0 < 1$ .

The proof of the theorem is completed.

**Theorem 4.3** Suppose that  $(H_1^*)$  and  $(H_3)$ – $(H_4)$  hold. Then there exists a constant  $\lambda^* > 0$  such that, for any  $0 < \lambda \leq \lambda^*$ , the boundary value problem (1.1) has at least one positive solution.

**Proof** Let  $\Omega_1 = \{x \in C[0, 1] : \|x\| < R_1\}$ , where  $R_1 = \max\{1, r\}$  and  $r = \int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)} g(s) ds$ . Choose

$$\lambda^* = \min\{1, R_1 [\int_0^1 (\alpha-1)q(s) [\max_{0 \leq z \leq R_1} f(s, z) + g(s)] ds]^{-1}\}.$$

Then for any  $x \in P \cap \partial\Omega_1$ , then  $\|x\| = R_1$  and  $x(s) - v(s) \leq x(s) \leq \|x\|$ , we have

$$\begin{aligned} \|Tx(t)\| &\leq \lambda \int_0^1 (\alpha-1)q(s) [f(s, [x(s) - v(s)]^*) + g(s)] ds \\ &\leq \lambda \int_0^1 (\alpha-1)q(s) [f(s, [x(s) - v(s)]^*) + g(s)] ds \\ &\leq \lambda \int_0^1 (\alpha-1)q(s) [\max_{0 \leq z \leq R_1} f(s, z) + g(s)] ds \\ &\leq R_1 = \|x\|. \end{aligned}$$

This implies

$$\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1.$$

On the other hand, choose a constant  $N > 0$  such that

$$\frac{\lambda N}{2(\alpha-1)} \min_{\theta_1 \leq t \leq \theta_2} k(t) \int_{\theta_1}^{\theta_2} s^\alpha (1-s)^\alpha ds \geq 1.$$

By the assumptions  $(H_3)$ , for any  $t \in [\theta_1, \theta_2]$ , there exists a constant  $B > 0$  such that

$$\frac{f(t, z)}{z} > N, \quad \text{namely } f(t, z) > Nz, \quad \text{for } z > B.$$



Choose  $R_2 = \max\{R_1 + 1, 2\lambda r, \frac{2(\alpha-1)(B+1)}{\min_{\theta_1 \leq t \leq \theta_2} \{t^{\alpha-1}(1-t)\}}\}$ , and let  $\Omega_2 = \{x \in C[0, 1] : \|x\| < R_2\}$ , then for any  $x \in P \cap \partial\Omega_2$ , we have

$$\begin{aligned} x(t) - v(t) &= x(t) - \lambda \int_0^1 G(t, s)g(s)ds \\ &\geq x(t) - \frac{t^{\alpha-1}(t-1)}{\alpha-1} \lambda \int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)} g(s)ds \\ &\geq x(t) - \frac{x(t)}{\|x\|} \lambda r \\ &\geq (1 - \frac{\lambda r}{R_2})x(t) \\ &\geq \frac{1}{2}x(t) \geq 0, \quad t \in [0, 1]. \end{aligned}$$

And then

$$\begin{aligned} \min_{\theta_1 \leq t \leq \theta_2} \{x(t) - v(t)\} &\geq \min_{\theta_1 \leq t \leq \theta_2} \left\{ \frac{1}{2}x(t) \right\} \geq \min_{\theta_1 \leq t \leq \theta_2} \left\{ \frac{t^{\alpha-1}(1-t)}{2(\alpha-1)} \|x\| \right\} \\ &= \frac{R_2 \min_{\theta_1 \leq t \leq \theta_2} t^{\alpha-1}(1-t)}{2(\alpha-1)} \geq B + 1 > B. \end{aligned}$$

$$\begin{aligned} \|Tx(t)\| &\geq \max_{0 \leq t \leq 1} \lambda \int_0^1 \Gamma(\alpha)k(t)q(s)[f(s, [x(s) - v(s)]^*) + g(s)]ds \\ &\geq \max_{0 \leq t \leq 1} \lambda \Gamma(\alpha)k(t) \int_0^1 q(s)f(s, [x(s) - v(s)]^*)ds \\ &\geq \lambda \min_{\theta_1 \leq t \leq \theta_2} \Gamma(\alpha)k(t) \int_{\theta_1}^{\theta_2} q(s)f(s, x(s) - v(s))ds \\ &\geq \lambda \min_{\theta_1 \leq t \leq \theta_2} \Gamma(\alpha)k(t) \int_{\theta_1}^{\theta_2} q(s)N(x(s) - v(s))ds \\ &\geq \lambda \min_{\theta_1 \leq t \leq \theta_2} \Gamma(\alpha)k(t) \int_{\theta_1}^{\theta_2} q(s)\frac{N}{2}x(s)ds \\ &\geq \frac{\lambda N}{2(\alpha-1)} \min_{\theta_1 \leq t \leq \theta_2} k(t) \int_{\theta_1}^{\theta_2} \Gamma(\alpha)q(s)s^{\alpha-1}(1-s)\|x\|ds \\ &\geq \frac{\lambda N}{2(\alpha-1)} \min_{\theta_1 \leq t \leq \theta_2} k(t) \int_{\theta_1}^{\theta_2} s^\alpha(1-s)^\alpha ds \|x\| \\ &\geq \|x\|. \end{aligned}$$

$$\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2.$$

Condition (2) of Krasnoesel'skii's fixed-point theorem is satisfied. So  $T$  has a fixed point  $x$  with  $r < \|x\| < R_2$  such that

$$\begin{aligned} -\mathbf{D}_{0+}^\alpha x(t) &= \lambda(f(s, [x(s) - v(s)]^*) + g(s)), \quad 0 < t < 1, n-1 < \alpha \leq n, \\ x^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ x(1) &= 0. \end{aligned}$$

Since  $r < \|x\|$ ,

$$\begin{aligned}
x(t) - v(t) &\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} \|x\| - \lambda \int_0^1 G(t,s)g(s)ds \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} \|x\| - \frac{t^{\alpha-1}(1-t)}{\alpha-1} \lambda \int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)} g(s)ds \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} \|x\| - \frac{t^{\alpha-1}(1-t)}{\alpha-1} \lambda r \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} r - \frac{t^{\alpha-1}(1-t)}{\alpha-1} \lambda r \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1} (1-\lambda)r \\
&> 0, \quad t \in (0,1).
\end{aligned}$$

Let  $u(t) = x(t) - v(t)$ , then  $u(t)$  is a positive solution of the boundary value problem (1.1).

The proof of the theorem is completed.

**Remark.** In Theorem 4.3,  $f$  may be singular at  $t = 0$  and 1.

Since condition  $(H_1)$  implies condition  $(H_1^*)$  and  $(H_4)$ , and from the proof of Theorem 4.2 and 4.3, we have immediately the following theorem:

**Theorem 4.4** Suppose that  $(H_1)$ -( $H_3$ ) hold. Then the boundary value problem (1.1) has at least two positive solutions for  $\lambda > 0$  sufficiently small.

In fact, let  $0 < \lambda < \min\{\bar{\lambda}, \lambda^*\}$ , then the boundary value problem (1.1) has at least two positive solutions  $u_1$  and  $u_2$ .

## 5 Examples

To illustrate the usefulness of the results, we give some examples.

**Example 1.** Consider the boundary value problem

$$\begin{aligned}
D_{0+}^\alpha u(t) + \lambda(u^a(t) + \frac{1}{(t-t^2)^{\frac{1}{2}}} \cos(2\pi u(t))) &= 0, \quad 0 < t < 1, \lambda > 0, \\
u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\
u(1) &= 0,
\end{aligned} \tag{5.1}$$

where  $a > 1$ . Then, if  $\lambda > 0$  is sufficiently small, then (5.1) has a positive solutions  $u$  with  $u(t) > 0$  for  $t \in (0,1)$ .

To see this we will apply Theorem 4.3 with

$$f(t, u) = u^a(t) + \frac{1}{(t-t^2)^{\frac{1}{2}}} \cos(2\pi u(t)), \quad g(t) = \frac{1}{(t-t^2)^{\frac{1}{2}}}.$$

Clearly

$$f(t, 0) = \frac{1}{(t-t^2)^{\frac{1}{2}}} > 0, \quad f(t, u) + g(t) \geq u^a(t) > 0, \quad \lim_{u \uparrow +\infty} \frac{f(t, u)}{u} = +\infty, \quad \text{for } t \in (0,1), u > 0.$$

Namely  $(H_1^*)$  and  $(H_2)$ -( $H_4$ ) hold. From  $\int_0^1 \frac{1}{(s-s^2)^{\frac{1}{2}}} ds = \frac{\pi}{2}$ , set  $R_1 = \frac{2(\alpha-1)\pi}{\Gamma(\alpha)}$ , then  $R_1 > \int_0^1 G(t,s)g(s)ds$ , we have

$$\begin{aligned}
\int_0^1 (\alpha-1)q(s) \left[ \max_{0 \leq z \leq R_1} f(s, z) + g(s) \right] ds &\leq \int_0^1 (\alpha-1)q(s) \left[ \left( \frac{2(\alpha-1)\pi}{\Gamma(\alpha)} \right)^a + \frac{2}{(s-s^2)^{\frac{1}{2}}} \right] ds \\
&\leq \frac{\alpha-1}{\Gamma(\alpha)} \left( \left( \frac{2(\alpha-1)\pi}{\Gamma(\alpha)} \right)^a + 2\pi \right)
\end{aligned}$$

and  $\lambda^* = \min\{1, \frac{\Gamma^a(\alpha)}{(\alpha-1)^a(2\pi)^{a-1} + \Gamma^a(\alpha)}\}$ . Now, if  $\lambda < \lambda^*$ , Theorem 4.3 guarantees that (5.1) has a positive solutions  $u$  with  $\|u\| \geq 2$ .

**Example 2.** Consider the boundary value problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) + \lambda(u^2(t) - \frac{9}{2}u(t) + 2) &= 0, \quad 0 < t < 1, \lambda > 0, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ u(1) &= 0, \end{aligned} \quad (5.2)$$

Then, if  $\lambda > 0$  is sufficiently small, then (5.2) has two solutions  $u_i$  with  $u_i(t) > 0$  for  $t \in (0, 1)$ ,  $i = 1, 2$ .

To see this we will apply Theorem 4.4 with (here  $0 < R_1 < 1 < R_2$  will be chosen below)

$$f(t, u) = u^2(t) - \frac{9}{2}u(t) + 2, \quad g(t) = 4.$$

Clearly

$$f(t, 0) = 2 > 0, \quad f(t, u) + g(t) \geq \frac{15}{16} > 0, \quad \lim_{u \uparrow +\infty} \frac{f(t, u)}{u} = +\infty, \quad \text{for } t \in (0, 1),$$

Namely (H<sub>1</sub>)-(H<sub>3</sub>) hold. Let  $\delta = \frac{1}{4}$  and  $\varepsilon = \frac{1}{4}$ , we may have

$$\bar{\lambda} = \frac{\varepsilon}{2c(\max_{0 \leq x \leq \varepsilon} f(t, x) + 4)} = \frac{1}{48c}, \quad (5.3)$$

where  $c = \int_0^1 (\alpha - 1)q(s)ds$ . Now, if  $\lambda < \bar{\lambda}$ , Theorem 4.2 guarantees that (5.2) has a positive solutions  $u_1$  with  $\|u_1\| \leq \frac{1}{4}$ .

Next, let  $R_1 = 5$ , we have

$$\int_0^1 (\alpha - 1)q(s) [\max_{0 \leq z \leq R_1} f(s, z) + g(s)] ds = \int_0^1 (\alpha - 1)q(s) [\frac{9}{2} + 4] ds = \frac{17c}{2}$$

and  $\lambda^* = \min\{1, \frac{10}{17c}\}$ . Now, if  $\lambda < \lambda^*$ , Theorem 4.3 guarantees that (5.2) has a positive solutions  $u_2$  with  $\|u_2\| \geq 5$ .

So, since all the conditions of Theorem 4.4 are satisfied, if  $\lambda < \min\{\bar{\lambda}, \lambda^*\}$ , Theorem 4.4 guarantees that (5.2) has two solutions  $u_i$  with  $u_i(t) > 0$  for  $t \in (0, 1)$ ,  $i = 1, 2$ .

**Example 3.** Consider the boundary value problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) + \lambda(u^a(t) + \cos(2\pi u(t))) &= 0, \quad 0 < t < 1, \lambda > 0, \\ u^{(j)}(0) &= 0, \quad 0 \leq j \leq n-2, \\ u(1) &= 0, \end{aligned} \quad (5.4)$$

where  $a > 1$ . Then, if  $\lambda > 0$  is sufficiently small, then (5.4) has two solutions  $u_i$  with  $u_i(t) > 0$  for  $t \in (0, 1)$ ,  $i = 1, 2$ .

To see this we will apply Theorem 4.4 with

$$f(t, u) = u^a(t) + \cos(2\pi u(t)), \quad g(t) = 2.$$

Clearly

$$f(t, 0) = 1 > 0, \quad f(t, u) + g(t) \geq u^a(t) + 1 > 0, \quad \lim_{u \uparrow +\infty} \frac{f(t, u)}{u} = +\infty, \quad \text{for } t \in (0, 1),$$

Namely (H<sub>1</sub>)-(H<sub>3</sub>) hold. By a similar way of example 2, if  $\lambda > 0$  is sufficiently small, Theorem 4.4 guarantees that (5.4) has two solutions  $u_i$  with  $u_i(t) > 0$  for  $t \in (0, 1)$ ,  $i = 1, 2$ .

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